PRINCIPLES OF ANALYSIS
LECTURE 10 - MONOTONE SEQUENCES

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1. Infinity

The extended real numbers are $\mathbb{R} \cup \{\pm\infty\}$.

If A is unbounded above, then $\sup A = \infty$.

If A is unbounded below, then $\inf A = -\infty$.

If $\lim a_n = \pm\infty$, we say that "diverges to +- infinity".

Arithmetic of infinity based on sequences can be developed.

We say that $\lim s_n > 0$ if $\{s_n\}_{n=1}^{\infty}$ converges to a positive real number, or if $\{s_n\}_{n=1}^{\infty}$ diverges to ∞.

Proposition 1. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\lim s_n > 0$. Then there exists $N \in \mathbb{Z}^+$ and $P > 0$ such that if $n \geq N$, then $s_n > P$.*

Proof. If $s_n \to +\infty$, this follows directly from the definition. Thus assume that $\lim s_n = L > 0$, and set $\epsilon = \frac{L}{2}$. Let N be so large that $|s_n - L| < \epsilon$ for $n \geq N$. Then for such n, $s_n > L - \epsilon$. Let $P = L - \epsilon$. □

Proposition 2. *Let $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be sequences of positive real numbers such that $\lim s_n = +\infty$ and $\lim t_n > 0$. Then*

(a) $\lim(s_n + t_n) = +\infty$;

(b) $\lim(s_n t_n) = +\infty$.

Proof. Let $M > 0$.

Since $\lim t_n > 0$, there exists $N_1 \in \mathbb{Z}^+$ and $P > 0$ such that if $n \geq N_1$ then $t_n > P$.

Since $\lim s_n = +\infty$, there exists $N_2 \in \mathbb{Z}^+$ such that if $n \geq N_2$ then $s_n > \frac{M}{P}$.

Set $N = \max N_1, N_2$; for $n \geq N$, we have

$$s_n t_n > \frac{M}{P} P = M.$$

□

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Proposition 3. *Let* $\{s_n\}_{n=1}^{\infty}$ *be a sequence of positive real numbers. Then*

$$\lim s_n = +\infty \Leftrightarrow \lim \frac{1}{s_n} = 0.$$

Proof. To show an if and only if statement, we show both directions.

($\Rightarrow$) Suppose that $\lim s_n = +\infty$. Let $\epsilon > 0$, set $M = \frac{1}{\epsilon}$. Since $s_n \to +\infty$, there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $s_n > M$. Then for $n \geq N$, we have $|\frac{1}{s_n} - 0| = \frac{1}{s_n} < \epsilon$.

($\Leftarrow$) Suppose that $\lim \frac{1}{s_n} = 0$. Let $M > 0$ and set $\epsilon = \frac{1}{M}$. Since $\frac{1}{s_n} \to 0$, there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $|\frac{1}{s_n} - 0| < \epsilon$. Since s_n is positive, this is the same as $\frac{1}{s_n} < \epsilon$, which implies that $s_n > M$. Thus $s_n \to +\infty$. □

2. Monotone Sequence

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

We say that $\{s_n\}_{n=1}^{\infty}$ is *increasing* if

$$m \leq n \Rightarrow s_m \leq s_n.$$

We say that $\{s_n\}_{n=1}^{\infty}$ is *decreasing* if

$$m \leq n \Rightarrow s_m \geq s_n.$$

We say that $\{s_n\}_{n=1}^{\infty}$ is *monotone* if it is either increasing or decreasing.

Proposition 4. *Let $\{s_n\}_{n=1}^{\infty}$ be a monotone sequence.*

(a) *If $\{s_n\}_{n=1}^{\infty}$ is bounded, then it converges.*
(b) *If $\{s_n\}_{n=1}^{\infty}$ is unbounded and increasing, then it diverges to $+\infty$.*
(c) *If $\{s_n\}_{n=1}^{\infty}$ is unbounded and decreasing, then it diverges to $-\infty$.*

Proof.

(a) Suppose that $\{s_n\}_{n=1}^{\infty}$ is bounded. Also assume that it is increasing; the proof for decreasing will be analogous. Let $S = \{s_n \mid n \in \mathbb{Z}^+\}$ be the image of the sequence, and set $u = \sup S$. Since S is bounded, $u \in \mathbb{R}$. Clearly $s_n \leq u$ for every $n \in \mathbb{Z}^+$. We show that $\lim s_n = u$.

Let $\epsilon > 0$. Since $u - \epsilon$ is not an upper bound for S, there exists $s \in S$ such that $u - \epsilon < s < u$. Now $s = s_N$ for some $N \in \mathbb{Z}^+$, and since $\{s_n\}_{n=1}^{\infty}$ is increasing, we have $u - \epsilon < s_n < u$ for every $n \geq N$. Thus $|s_n - u| < \epsilon$ for $n \geq N$; this shows that $s_n \to u$.

(b) Let $M > 0$. □

Proposition 5. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Set*

$$u_N = \sup\{s_n \mid n \geq N\}$$

and

$$v_N = \inf\{s_n \mid n \geq N\}.$$

Then $\{u_n\}_{n=1}^{\infty}$ is a decreasing sequence and $\{v_n\}_{n=1}^{\infty}$ is an increasing sequence.

Proof. □

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Define

$$\limsup s_n = \lim_{N \to \infty} \sup\{s_n \mid n \geq N\}$$

and

$$\liminf s_n = \lim_{N \to \infty} \inf\{s_n \mid n \geq N\}.$$

Proposition 6. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{s_n\}_{n=1}^{\infty}$ converges if and only if $\liminf s_n = \limsup s_n$, in which case $\liminf s_n = \lim s_n = \limsup s_n$.*

3. Cluster Points of Sequences

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, and let $c \in \mathbb{R}$. We say that c is a *cluster point* of $\{s_n\}_{n=1}^{\infty}$ if

$$\forall \epsilon > 0 \; \forall N \in \mathbb{Z}^+ \; \exists n \geq N \; \ni \; |s_n - c| < \epsilon.$$

Proposition 7. *Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Let C be the set of cluster points of $\{s_n\}_{n=1}^{\infty}$. Then*

- **(a)** $\limsup s_n \in C$;
- **(b)** $\liminf s_n \in C$;
- **(c)** $\sup C = \limsup s_n$;
- **(d)** $\inf C = \liminf s_n$.

Proof. Since $\{s_n\}_{n=1}^{\infty}$ is bounded, $\limsup s_n$ and $\liminf s_n$ exist as real numbers. Let $s = \limsup s_n$; we will prove **(a)** and **(c)**, the proofs for **(b)** and **(d)** being analogous.

For **(a)**, suppose not; then $s \notin C$. That is, there exists $\epsilon > 0$ and $N \in \mathbb{Z}^+$ such that $|s_n - s| > \epsilon$ for all $n \geq N$. Now either there exists $n \geq N$ such that $s_n > s + \epsilon$, or for every $n \geq N$, $s_n < s - \epsilon$.

In the first case, s cannot be an upper bound for S, a contradiction. In the second case, $s - \epsilon$

□

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